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ON SURFACES IN DIGITAL TOPOLOGY

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Abstract: *In [Ayala∞] a new framework for digital topology has been proposed. This framework offers the possibility of transfering, in an easy way, definitions, statements and proofs from continuous topology to digital topology (see details in §2). In particular, it provides a straightforward definition of n-dimensional digital manifold.*

In this paper we prove that the class of digital 2-manifolds without boundary in the grid $\mathbb{Z}^3$ agrees with the class of $(26,6)$-surfaces defined by Kong-Roscoe and other authors ([Morgenthaler81],[Reed84],[Kong85]). As a consequence, the separation theorem for digital surfaces stated in [Morgenthaler81] and [Reed84] is obtained.

Keywords: digital topology, digital surfaces

1 Introduction

In [Ayala∞] a new framework for digital topology has been proposed. In this framework, the pixels on a computer screen are represented by means of a polyhedral complex. Associated to this polyhedral complex, three different models (the *logical, conceptual* and *continuous* models) are defined and these models allow us to transfer, in an easy way, definitions, statements and proofs from continuous topology to digital topology (see details in §2). In particular, this framework provides a straightforward definition of n-dimensional digital manifold in such a way that closed digital 1-manifolds in the grid $\mathbb{Z}^2$ correspond to the well-known digital Jordan curves in Rosenfeld's sense ([Rosenfeld79]). In this paper we prove that the class of digital 2-manifolds without boundary in the grid $\mathbb{Z}^3$ agrees with the class of $(26,6)$-surfaces defined by Kong-Roscoe and other authors ([Morgenthaler81],[Reed84],[Kong85]). As a consequence, the separation theorem for digital surfaces stated in [Morgenthaler81] and [Reed84] is obtained.

It must be pointed out that Kong and Roscoe actually define (α,β)-surfaces for $\alpha,\beta \in \{6,18,26\}$. Of these surfaces, however, all except $(18,6)$, $(6,18)$, $(26,6)$, and $(6,26)$ are usually discarded on the grounds that their restrictions on the grid $\mathbb{Z}^2 \times \{0\} \subset \mathbb{Z}^3$ produce the paradoxical 8 or 4-adjacency relations for $\mathbb{Z}^2$. Our characterization suggests that, among the non-paradoxical (α, β)-surfaces, the $(26, 6)$-surfaces are more likely to provide a theory for $\mathbb{Z}^3$ which as nearly as possible replicates that of $\mathbb{R}^3$.

2 Digital Topology and polyhedral complexes

In this paper K will denote a locally finite and homogeneously n-dimensional polyhedral complex. Namely, K is a complex of convex cells (*polytopes*) such that each polytope is face of a finite number (non-zero) of n-polytopes. If $\sigma \in K$, the boundary of the polytope σ is the set $\partial\sigma$ union of its faces. The interior of σ is the set $\mathring{\sigma} = \sigma - \partial\sigma$.

If $|K|$ denotes the underlying polyhedron of K, a *centroid-map* is a map $c : K \to |K|$ such that $c(\sigma) \in \mathring{\sigma}$. The point $c(\sigma)$ is called *centroid* of σ and the pair (K, c) is called *device model*. Given (K, c), we define an undirected graph $\mathcal{L}_{(K,c)}$ (or simply $\mathcal{L}_K$) whose vertices are the centroids of n-polytopes in K and two vertices are adjacents in $\mathcal{L}_K$ if their corresponding n-polytopes intersect. The graph $\mathcal{L}_K$ is called the *logical model* of K.

The digraph $\mathcal{C}_K$, called *conceptual model* of K, is defined as follows. Its vertices are those of $\mathcal{L}_K$ and, in addition, the centroids $c(\sigma)$ such that σ is the intersection of two or more n-polytopes of K. The directed edges are pairs $(c(\tau), c(\sigma))$ with τ face of σ. If no confusion arises, we identify each centroid $c(\sigma)$ with the corresponding polytope σ, and the device model (K, c) with the polyhedral complex K.

A *digital object* in a device model K is a subset O of the set of centroids of n-polytopes of K. From the subgraph of $\mathcal{L}_K$ generated by O, denoted $i(O)$, we define the subgrah $pi(O)$ of $\mathcal{C}_K$ generated by the vertices of $i(O)$ together with the centroids of polytopes which are the intersection of two or more n-polytopes associated to vertices of $i(O)$.

Example 1 *In this paper we shall deal with the standard cubical decomposition of $\mathbb{R}^n$, K^n. That is, the device model determined by the collection of unit n-cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes and whose centers are the points of $\mathbb{Z}^n \subset \mathbb{R}^n$. The centroid-map associates to each cube σ its center $c(\sigma)$. Thus a digital object in K^n corresponds to a subset of $\mathbb{Z}^n$.*

A subset $C \subset O$ is called a *component* of the digital object O if $i(C)$ is a connected component of $i(O)$. On the other hand, if O^c denotes the complement of O in the set of n-polytopes of K, a subset $D \subset O^c$ is called a *DIG-component* of O^c if D is the set of centroids of O^c which belong to a connected component of $\mathcal{C}_K - pi(O)$.

The *continuous analogue* $|\mathcal{A}_O|$ of a digital object O is the underlying polyhedron of the order complex $\mathcal{A}_O$ associated to the graph $pi(O)$. That is, $\langle x_0, x_1, \ldots, x_m\rangle$ is a m-simplex of $\mathcal{A}_O$ if $x_0x_1 \ldots x_m$ is a directed path in the digraph $pi(O)$. This simplicial complex admits a polyhedral inmersion in K and thus $|\mathcal{A}_O|$ can be considered a subpolyhedron of $|K|$. It is easy to verify that there exists a bijective map between the set of DIG-components of the complement O^c and the set of connected components of the topological space

$|\mathcal{A}_K| - |\mathcal{A}_O|$. In fact, each DIG-component $D \subset O^c$ is determined by the n-polytopes in $pi(O^c) \cap A$, where A is a connected component of $|\mathcal{A}_K| - |\mathcal{A}_O|$.

A digital object O is a *digital manifold* if $|\mathcal{A}_O|$ is a combinatorial manifold. If $|\mathcal{A}_O|$ only is a topological manifold then O is called a *weak digital manifold*. In the standard cubical decomposition K^2 of $\mathbb{R}^2$, a digital object O is a closed digital 1-manifold if and only if O is a 8-curve in Rosenfeld's sense (see [Ayala∞] and [Rosenfeld79]). In §3 the relationship between the digital 2-manifolds in K^3 and the notion of surface due to Kong-Roscoe and other authors ([Morgenthaler81],[Reed84],[Kong85]) is studied.

Let us state here a version of a Generalized Digital Jordan Theorem, which can be easily proved by using the corresponding continuous result (III.11.17 in [Massey78], for example) and our previous definitions.

Theorem 2 (Generalized Digital Jordan Theorem) *Let K be a polyhedral complex such that $|K| = \mathbb{R}^n$. If a digital object O in K is a weak digital $(n-1)$-manifold without boundary, then the complement O^c is divided in two DIG-components. Moreover, if O is finite then one DIG-component is finite.*

3 Digital 2-manifolds and $(26, 6)$-surfaces

Let K^3 be the standard cubical decomposition of $\mathbb{R}^3$. Given a cube $\sigma \in K^3$, let $N(\sigma)$ denote the set of cubes in K^3 which meet σ (σ itself included). A cube $\mu \in N(\sigma)$ is said to be a 26-*neighbour* of σ. The cube μ is said to be a 18-*neighbour* of σ if $\dim \sigma \cap \mu \geq 1$. And μ is said to be a 6-*neighbour* of σ if $\dim \sigma \cap \mu \geq 2$. Let $N^\beta(\sigma)$ denote the set of β-neighbours of σ ($\beta = 6, 18, 26$). Clearly $N^6(\sigma) \subset N^{18}(\sigma) \subset N^{26}(\sigma) = N(\sigma)$.

Let O be a digital object in K^3. Two cubes $\sigma, \tau \in O$ are said to be *β-connected in O* ($\beta = 6, 18, 26$) if there exists a sequence of cubes $\sigma = \sigma_1, \ldots, \sigma_k = \tau$ in O such that σ_i is β-neighbour of σ_{i+1} for $1 \leq i \leq k-1$. A digital object O is said to be *β-connected* if any two cubes $\sigma, \tau \in O$ are β-connected in O. A *β-component* of O is a maximal β-connected subset of O. Notice that O is 26-connected if and only if O is connected in the digital sense (see §2). The DIG-components of the complement O^c are characterized in the following result.

Theorem 3 *Given a digital object O in K^3, the DIG-components of its complement O^c are exactly the 6-components of O^c.*

We are now considering a new polyhedral decomposition of $\mathbb{R}^3$, denoted $K^3(\mathbb{Z}^3)$, consisting of unit cubes with vertices in $\mathbb{Z}^3$. To avoid misunderstandings, we keep the terminology *cube* for the 3-polytopes of K^3 and we call $\mathbb{Z}^3$-*cells* the closed cubes in $K^3(\mathbb{Z}^3)$. Given a cube $\sigma \in K^3$, the *cuboid* $\mathbb{Z}^3(\sigma)$ of σ is the union of all the $\mathbb{Z}^3$-cells whose vertices correspond to centers of cubes in $N(\sigma)$. Given a $\mathbb{Z}^3$-cell A and a digital object O in K^3, the set $A \cap O$ will be called the *configuration* of O in A (the points in $A \cap O$ will be marked in figures by "•").

Associated with some specific configurations, Kong and Roscoe [Kong85] define certain 2-dimensional polyhedra called *plates*. For each adjacency $\beta =$

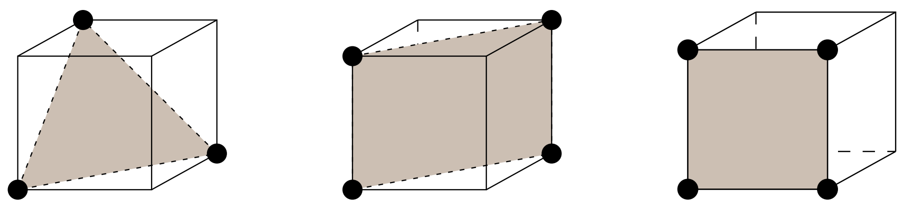

Figure 1.

$6, 18, 26$ a family $\mathbb{F}_\beta$ of admissible plates is given. Here we show only the family $\mathbb{F}_6$ which consists of the plane regions displayed in Figure 1, associated to the corresponding configurations (up to rotation or reflection). See [Kong85] for a description of $\mathbb{F}_{18}$ and $\mathbb{F}_{26}$.

A *plate cycle* at a vertex $c(\sigma) \in \mathbb{Z}^3$ is a sequence $\{\pi_i; 0 \leq i \leq k\}$ of distinct plates such that

(i) There is a sequence $\{e_i; 0 \leq i \leq k\}$ in which e_i and e_{i+1} are distinct edges of π_i $(0 \leq i < k)$, e_0 and e_n are distinct edges of π_k, and $c(\sigma)$ is a vertex of each e_i.

(ii) If $i \neq j$ then $\pi_i \cap \pi_j$ is the union of *(a)* a number (possibly zero) of straight line segments each of which is an edge of both plates, and *(b)* a set of vertices of both plates.

(iii) Any edge of a π_i is an edge of at most one other π_j.

Using the concept of $\mathbb{F}_6$-plate, Kong and Roscoe give the following characterization of the $(26, 6)$-surfaces. This characterization is stated here as a definition. Namely,

Definition 4 *(Prop. 12 in [Kong85]) Let O be a digital object in K^3. The digital object O is said to be a* $(26, 6)$-surface *if the following three conditions hold:*

(i) No configuration of O contains more than four points, and only the configurations with four points in Figure 1 are possible.

(ii) The set of $\mathbb{F}_6$-plates of O which contain the point $c(\sigma)$, $\mathbb{F}_6(O)(\sigma)$, defines a plate cycle at $c(\sigma)$.

(iii) If $\tau \in N(\sigma) \cap O$ then $c(\tau)$ is the vertex of a plate in $\mathbb{F}_6(O)(\sigma)$.

It is not difficult to prove:

Lemma 5 *The plates in $\mathbb{F}_6(O)(\sigma)$ define a polyhedral decomposition of a 2-disk contained in the cuboid $\mathbb{Z}^3(\sigma)$. Furthermore the 2-cells of this decomposition are triangles or rectangles.*

From this result we obtain:

Proposition 6 *Let O be a $(26, 6)$-surface. Then the family of all $\mathbb{F}_6$-plates of O defines a polyhedral decomposition $\mathcal{P}(O)$ of a 2-manifold without boundary. Moreover, $\mathcal{A}_O$ is a triangulation of $\mathcal{P}(O)$.*

Corollary 7 *Each $(26, 6)$-surface is a digital 2-manifold without boundary.*

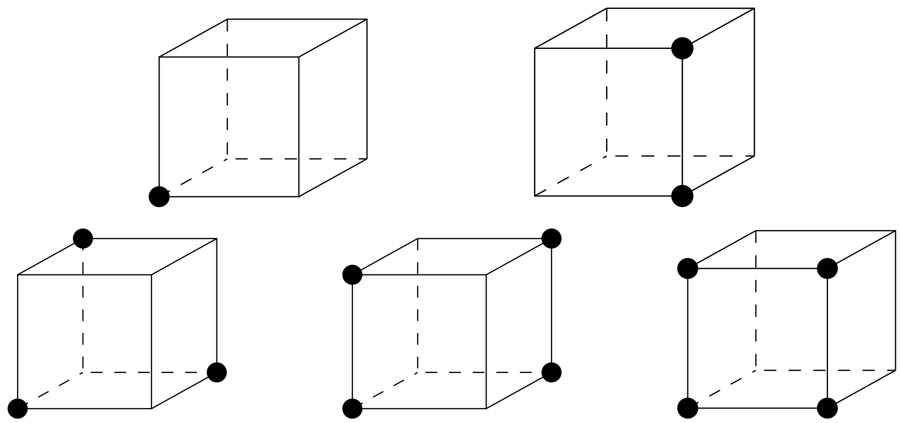

Figure 2.

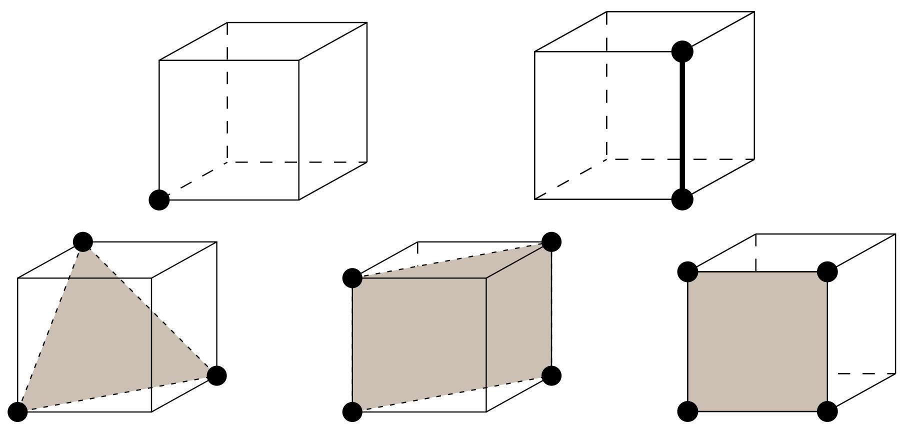

Figure 3.

In order to prove the converse, the following result is needed.

Proposition 8 *Let O be a digital object in K^3 such that $\mathcal{A}_O$ is a connected 2-complex in which each edge is the intersection of exactly two triangles (in other words, O is a digital 2-pseudomanifold without boundary). Then no configuration of O contains more than four points and only the configurations in Figure 2 (after a suitable rotation or reflection) can appear.*

Given a $\mathbb{Z}^3$-cell A and a digital object O in K^3, the *trace* of O in A is the subpolyhedron of $|\mathcal{A}_O|$ contained in A. Then Proposition 8 yields

Proposition 9 *If O is a digital 2-pseudomanifold without boundary in K^3 only the traces in Figure 3 (after a suitable rotation or reflection) can appear. In particular, conditions 4(i) and 4(iii) hold for O.*

Proposition 10 *If O is a digital 2-manifold without boundary in K^3 then condition 4(ii) also holds for O.*

As a consequence of these two propositions we obtain:

Corollary 11 *Each digital* 2*-manifold without boundary in* K^3 *is a* $(26, 6)$*-surface.*

So, from corollaries 7 and 11 the next characterization follows.

Theorem 12 *A digital object* O *in* K^3 *is a digital* 2*-manifold without boundary if and only if* O *is a* $(26, 6)$*-surface.*

This characterization, Theorem 3 and the Generalized Digital Jordan Theorem 2 allow us to recover, without any new proof, Reed's digital separation theorem (Thm. 1 in [Reed84]) and the main theorem in [Morgenthaler81]:

Theorem 13 *A* $(26, 6)$*-surface divides* $\mathbb{Z}^3$ *in two* 6*-components.*

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